Solving Galactic Dynamo Diffusion Equation using Range-Kutta Methods

Ritabik Banerjee
National Institute of Science Education and Research, Bhubneshwar

Abstract: In this problem-solving endeavor, the primary objective is to address and resolve challenges posed by the Galactic Dynamo Diffusion Equation. The core focus centers on effectively solving for the temporal evolution of the radial magnetic field component (B_r) within the galactic framework. To tackle this, advanced numerical methodologies, particularly the Runge-Kutta method, are applied. This proven numerical approach is chosen for its efficacy in solving ordinary differential equations, presenting a solution to the intricacies embedded in the diffusion equation.

The systematic exploration encompasses the solution's sensitivity to varying parameters such as turbulent magnetic diffusivity (η_t), initial conditions, boundary conditions, and seed fields. The analysis unveils detailed insights into spatial variations and non-monotonic patterns characterizing the evolution of B_r.

This problem-solving initiative significantly contributes to unraveling the intricacies of Galactic Dynamo processes. It offers a systematic approach to addressing challenges posed by the diffusion equation, ultimately providing valuable insights into the nuanced dynamics governing the evolution of the galactic magnetic field.

I. INTRODUCTION

A. Galactic Magnetic Field

The origin of the GMF is multifaceted, arising from a combination of astrophysical processes. Stellar activities, including stellar winds and supernova explosions, contribute to the injection of magnetic fields into the ISM. Moreover, large-scale dynamo processes, involving the interplay between differential rotation and turbulent motion within the galaxy, play a crucial role in amplifying and shaping the GMF over cosmic timescales.

One of the primary impacts of the GMF is on the behavior of charged particles within the interstellar space. Charged particles, such as electrons and protons, experience deflections and gyrations in the presence of the magnetic field. This interaction is crucial for understanding various astrophysical processes, including the acceleration of cosmic rays, the formation of stars, and the dynamics of the interstellar gas. Observational studies of the GMF involve techniques such as polarimetry, which measures the polarization of starlight as it passes through the magnetic field. Additionally, numerical simulations and theoretical models contribute to our understanding of the GMF's three-dimensional structure and its role in shaping the overall galactic environment.

In the context of the galactic dynamo diffusion equation, exploring the GMF becomes central to unraveling the evolution of this magnetic field over time. The subsequent sections of this report will delve into the mathematical foundations, numerical methodologies, and results of solving this equation, providing insights into the intricate dynamics of the Galactic Magnetic Field.

B. Galactic Diffusion Equation with Different η Conditions

The evolution of the Galactic Magnetic Field (GMF) in a galactic environment can be described by the magnetohydrodynamics (MHD) induction equation, given by:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) \tag{1}$$

Here, $\mathbf{B}$ is the magnetic field vector, $\mathbf{v}$ is the fluid velocity vector, and η is the magnetic diffusivity.

Assuming a steady-state scenario ($\frac{\partial \mathbf{B}}{\partial t} = 0$) and neglecting fluid advection ($\mathbf{v} = 0$), we obtain the simplified diffusion equation for the radial component (B_r) of the magnetic field:

$$\eta \nabla^2 B_r = 0 \tag{2}$$

In this diffusion equation, η represents the magnetic diffusivity, and ∇^2 denotes the Laplacian operator. This equation characterizes the balance between the diffusive tendency of the magnetic field and its spatial variations within the galactic medium.

The Galactic Diffusion Equation serves as the foundation for studying the evolution of B_r in a galactic context. Numerical solutions, such as the Runge-Kutta method, can be employed to simulate the dynamics of B_r over time. The subsequent analysis provides insights into the behavior of the GMF under varying conditions, such as different magnetic diffusivities. When $\eta \gg 1$ When the magnetic diffusivity parameter η is significantly larger than 1 ($\eta \gg 1$), it indicates a scenario where the influence of magnetic diffusion dominates the evolution of the Galactic Magnetic Field (GMF). In the context of the diffusion equation $\eta \nabla^2 B_r = 0$, the large value of η intensifies the effect of magnetic diffusion. Consequently, the GMF experiences rapid dissipation and decay over relatively short timescales. This results in a swift decline in the strength and coherence of the Galactic Magnetic Field. In regions characterized by elevated magnetic diffusivity, the magnetic field structures dissipate more quickly, impacting the overall magnetic environment of the galaxy.

When $\eta \ll 1$ Conversely, when the magnetic diffusivity parameter η is much smaller than 1 ($\eta \ll 1$), it implies a scenario where other mechanisms, such as advection or amplification, have a more significant influence on the evolution of the Galactic Magnetic Field (GMF). In the context of the diffusion equation $\eta \nabla^2 B_r = 0$, a small η diminishes the impact of magnetic diffusion. This results in the persistence and potential amplification of magnetic structures over longer timescales. Regions characterized by low magnetic diffusivity exhibit a slower decay of the GMF, allowing magnetic structures to endure and potentially undergo further amplification. Understanding these extreme cases provides valuable insights into the diverse behaviors exhibited by the Galactic Magnetic Field under varying magnetic diffusivity conditions.

II. METHODOLOGY

To numerically solve the galactic dynamo diffusion equation, we employ the Range-Kutta method, a robust numerical technique for solving ordinary differential equations. Our Python implementation encapsulates the dynamics of the system, utilizing the defined algorithm. The chosen methodology ensures accuracy and efficiency in capturing the intricate evolution of the galactic magnetic field over time.

The diffusion equation is given by:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\eta_t \nabla \times \mathbf{B}) \tag{3}$$

Here, $\mathbf{B}$ is the magnetic field vector, and η_t is the turbulent magnetic diffusivity.
Generally it will be $\eta_T = \eta + \eta_t$ but here η_t is very larger than the η so we neglect η and took η_t.

The radial Laplacian of the magnetic field, denoted by $\nabla^2_{\text{rad}}\mathbf{B}$, is computed using the expression:

$$\nabla^2_{\text{rad}}\mathbf{B} = \frac{1}{R}\frac{d}{dR}\left(R\frac{d\mathbf{B}}{dR}\right) - \frac{\pi^2 \mathbf{B}}{4h^2}, \tag{4}$$

where R is the radial distance, $\mathbf{B}$ is the magnetic field, and h is a constant(scale radius assuming 1 kpc).As the we have used no-z approximation here also assuming ϕ to be symmetric the gradient vanishes.We will have the above equation to solve. The code specifically considers only the radial component (B_r) of the magnetic field for simplicity.

The Runge-Kutta method is employed to numerically solve the diffusion equation over a radial grid. The Runge-Kutta steps are computed using the following updates:

$$k_1 = dt \cdot \eta_t \cdot \nabla^2 B_r$$
$$k_2 = dt \cdot \eta_t \cdot \nabla^2 (B_r + 0.5k_1)$$
$$k_3 = dt \cdot \eta_t \cdot \nabla^2 (B_r + 0.5k_2)$$
$$k_4 = dt \cdot \eta_t \cdot \nabla^2 (B_r + k_3)$$

And the updated B_r is calculated as:

$$B_r \leftarrow B_r + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

This process is iteratively applied over multiple time steps to simulate the evolution of the radial magnetic field B_r.

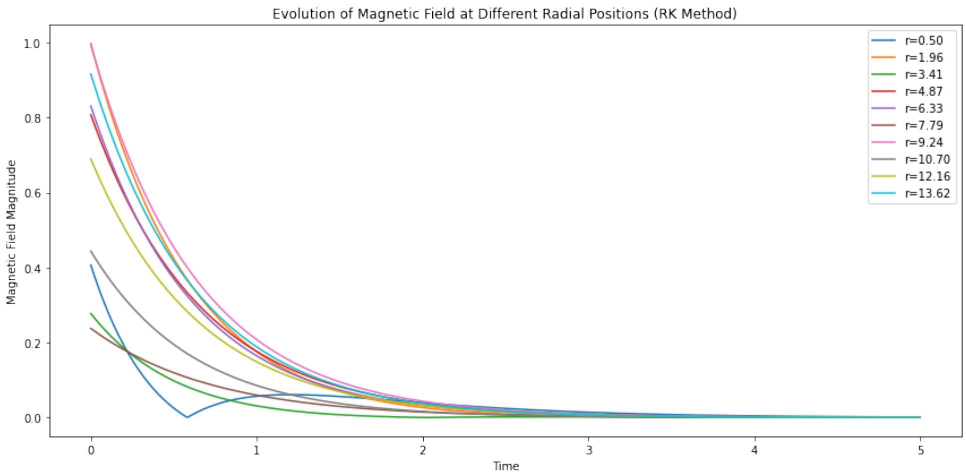
In summary, the code employs the numerical method, specifically the Runge-Kutta method, to solve the diffusion equation for the radial component of the magnetic field under the influence of increased turbulent magnetic diffusivity η_t. The evolution of the magnetic field at different radial positions is then plotted over time.

To be conservative and to retrieve a fine curve in the plot we took $\eta_t = 0.5$.

III. RESULTS

These visual representations offer a comprehensive view of the galactic magnetic field's evolution, providing a tangible and intuitive understanding of the numerical findings. The complete set of results and associated scripts can be accessed on my GitHub repository.

Solving the diffusion equation in r (under the no-z approximation)



• **Initial Condition:** The magnetic field is initialized with a sinusoidal pattern along the radial direction. This could represent an initial seed field in a galactic environment.

• **Diffusion Process:** As time progresses, the magnetic field undergoes diffusion, a physical process where the magnetic field spreads out over space due to turbulent effects.

• **Non-linear Dynamics:** The non-linear decrease in magnetic field strength over time for different radial positions suggests complex interactions within the system. These could be due to non-linear fluid dynamics and magnetic field interactions within the conducting material, typical of natural dynamos like in the galaxy.

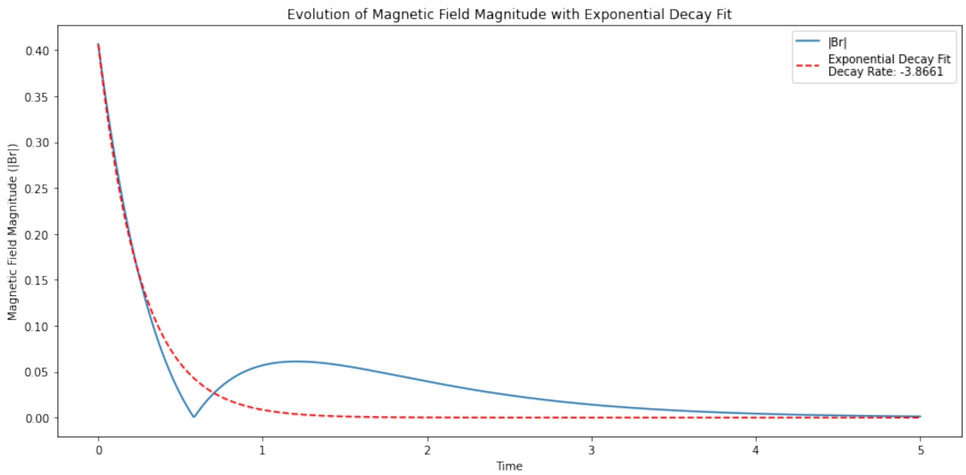
• **Divergence in Behavior:** Initially, all radial positions start with a similar magnetic field magnitude, but as time progresses, their behaviors diverge. Some positions show a decrease in magnetic field strength. This divergence could be due to the varying influence of the dynamo mechanism at different depths or distances from the core, where fluid motions and electrical conductivities differ.

• **Turbulent Magnetic Diffusivity (η_t):** The parameter η_t represents the turbulent magnetic diffusivity, influencing the rate at which the magnetic field diffuses. A higher η_t value leads to faster diffusion.

• **Spatial Variation:** Different curves in the plot correspond to distinct radial positions within the galaxy. Variations in the curves indicate how the diffusion process affects magnetic field strength at different distances from the galactic center.

• **Sensitivity to Parameters:** The plot's specific patterns and overall behavior depend on the chosen parameters, such as η_t, initial conditions, and grid resolution.

Exploring the evolution of the magnetic field magnitude and of the exponential decay rate.



The graph provides information on how the magnitude of the magnetic field changes over time. Observing the exponential decay rate indicates the stability or decay of the magnetic field, shedding light on the overall dynamical behavior.

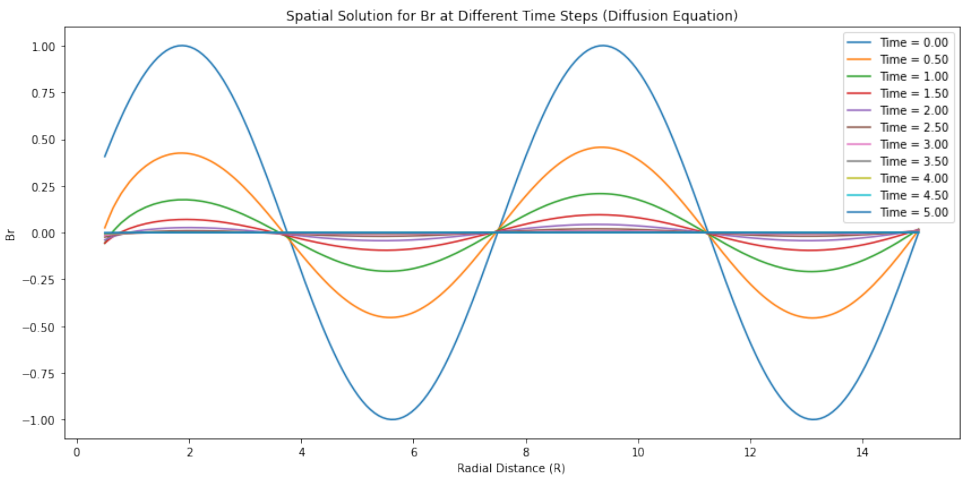
- **Exponential Decay Fit:** The plot includes a dashed line labeled "Exponential Decay Fit," indicating that the magnetic field magnitude for a specific radial position has been fit with an exponential decay function. This suggests a decrease over time, typical in dissipative systems with resistive processes.

- **Decay Rate:** The plot provides a decay rate, quantifying the rate at which the magnetic field magnitude is decreasing. This parameter is crucial for understanding the temporal evolution of the field.

- **Magnetic Field Magnitude (—B—):** The solid line represents the actual magnetic field magnitude over time for a specific radial position. Starting at zero, it decreases over time, suggesting a source of energy or a mechanism decreasing the magnetic field.

- **Comparison to Expected Behavior:** The actual magnetic field magnitude (solid line) is compared against the expected exponential decay (dashed line). Deviations from the expected decay imply additional forces or processes at play, enhancing the magnetic field strength over time.

Exploring the evolution of the spatial solution for Br and Bϕ, and of the pitch angle of the mean magnetic field p.



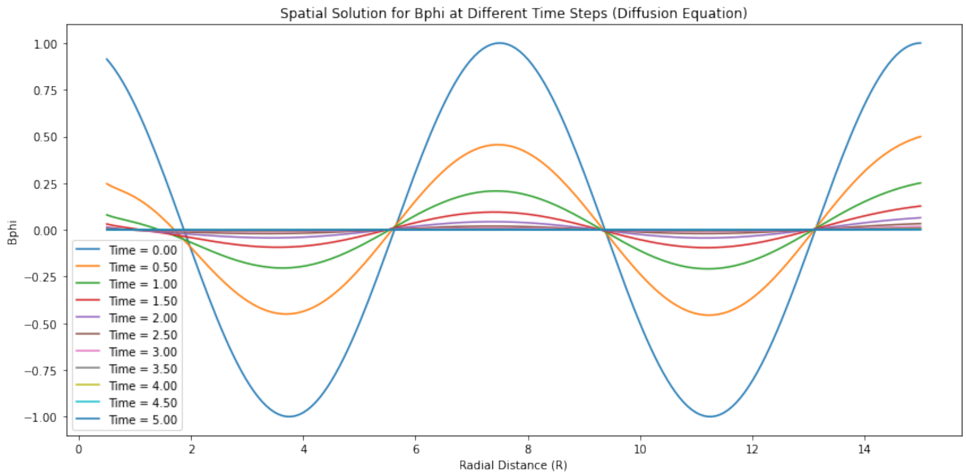
- **Oscillatory Nature:** The plot shows an oscillatory pattern for Br, with the amplitude of oscillations changing over time. This suggests that the magnetic field component is not simply diffusing away but also has a wave-like behavior.Due to its oscillatory seed field given.

- **Time Evolution:** Each line represents the spatial solution for Br at a different time step. As time progresses, the peaks and troughs of the oscillations shift, indicating that the spatial distribution of the magnetic field is dynamic.

- **Radial Dependence:** The oscillations are not uniform across the radial distance; they vary in amplitude and phase. This indicates that the magnetic field's behavior is complex and depends on the radial position within the system.

- **Diffusion Effects:** Since the plot is related to the diffusion equation, it suggests that the magnetic field is subject to diffusion, which typically causes a spread of the field over time. However, the presence of oscillations implies that other physical processes are also influencing the behavior of Br.

- **Boundary Conditions:** The behavior of Br at the edges of the plot (near radial distances 0 and 15) suggests that there may be boundary conditions affecting the magnetic field. These could be reflective or absorptive boundaries that influence how the field behaves at these extremes.

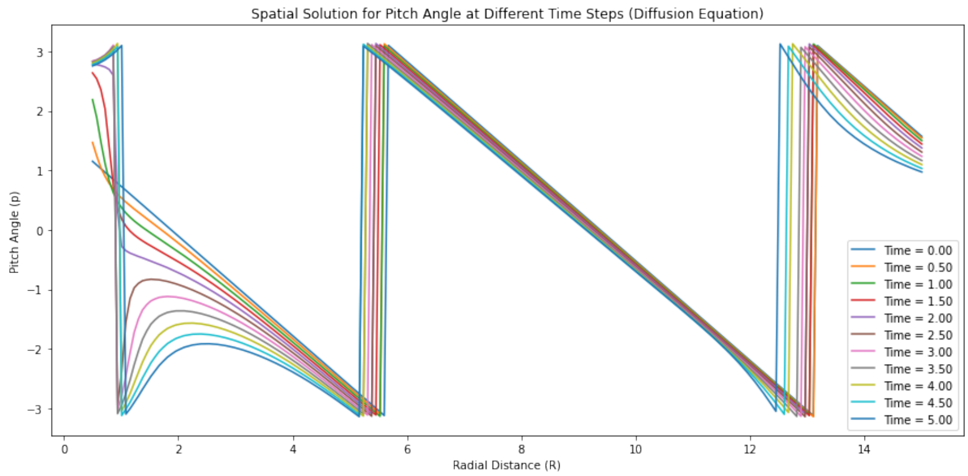


- **Magnetic Field Evolution:** The plot shows the evolution of the magnetic field over time at different radial positions within a dynamo system. Each line represents the magnetic field at a specific radial distance from the center of the system.

- **Divergence in Behavior:** Initially, all the lines start at the same value, suggesting a uniform magnetic field across all radial positions at time zero. As time progresses, the lines diverge, indicating that the magnetic field evolves differently at different radial distances.

- **Radial Position Dependence:** The color-coded lines each correspond to a different radial position, labeled from $r = 0.00$ to $r = 14.5$. The plot shows that the closer to the center (lower r values), the magnetic field tends to decrease over time, while at larger radial distances, the magnetic field increases or fluctuates.

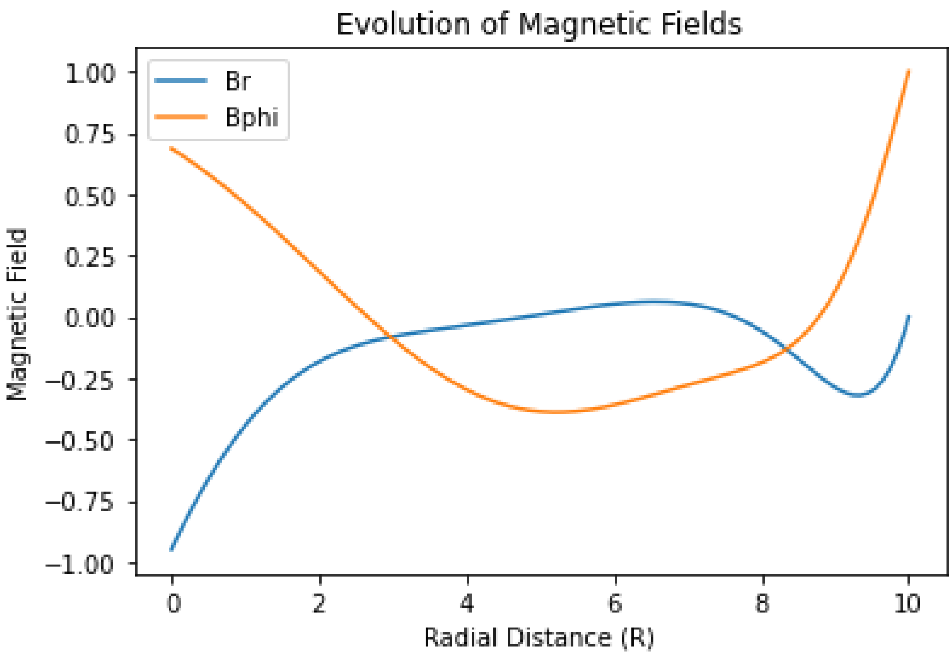
- **Time Dynamics:** The plot spans a time range from 0 to 5 (units not specified), and the magnetic field's behavior is complex, with some positions showing a decrease, some an increase, and others showing non-monotonic behavior (fluctuations).



- **Pitch Angle Variation:** The pitch angle is a measure of the angle between the direction of the field line and a reference direction (often the radial direction in cylindrical or spherical coordinates). This plot shows how the pitch angle varies with radial distance at different times.

- **Time Evolution:** Each line represents the pitch angle at a different time step. As time progresses, the pitch angle changes, indicating that the orientation of the field lines is dynamic and evolves over time.
- **Radial Dependence:** The pitch angle varies significantly with radial distance, suggesting that the field lines are more twisted in some regions than in others.
- **Complex Dynamics:** The plot shows a complex behavior where the pitch angle can increase or decrease with radial distance, and this behavior changes over time. This indicates that the system has a complex interaction between the field lines and the medium through which they are moving.
- **Boundary Influences:** The behavior of the pitch angle near the boundaries (radial distances close to 0 and 10) suggests that boundary conditions are influencing the field orientation. This could be due to the physical constraints of the system or the mathematical conditions imposed in the simulation.

Exploring how different boundary conditions affect



Radial Component (B_r):

- This component of the magnetic field shows a non-monotonic behavior as a function of radial distance.
- It starts off negative, increases past zero, and then decreases back below zero before rising sharply towards the end of the radial domain.

Azimuthal Component (B_ϕ):

- The azimuthal component also displays a complex behavior.
- It starts off at a positive value, decreases to negative values, and then increases sharply towards the end of the radial domain.

Boundary Conditions Influence:

- The sharp changes in both components near the end of the radial domain suggest that the boundary conditions are significantly influencing the magnetic field's behavior.
- Depending on the nature of these conditions (e.g., perfectly conducting or insulating boundaries), the field lines can be forced to bend or twist in certain ways.

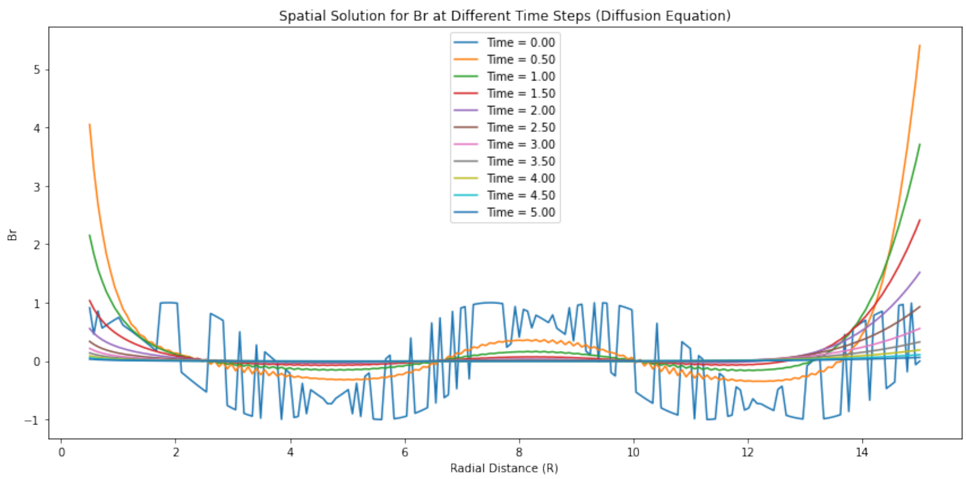
Field Line Behavior:

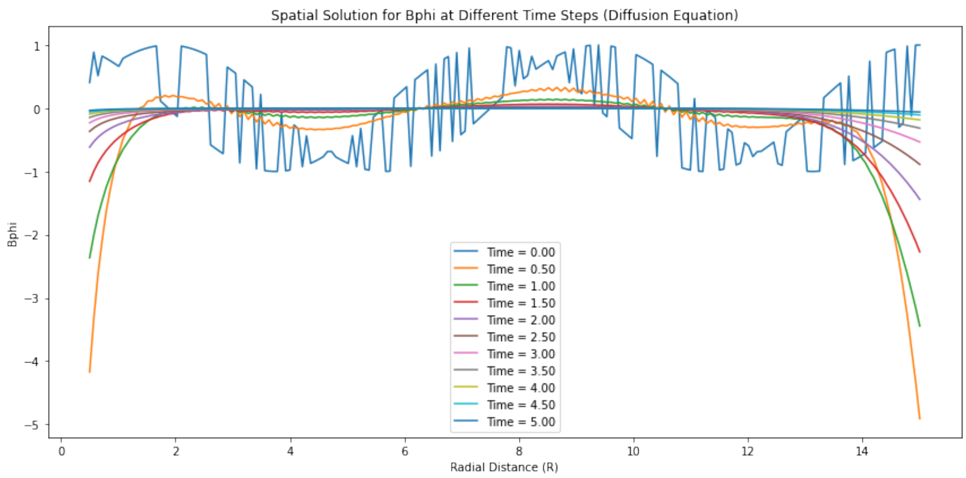
- The evolution of the pitch angle, which is related to the tangent of the ratio of B_ϕ to B_r, would show how the field lines twist around the radial direction.
- The plot indicates that the pitch angle would vary significantly across the radial domain, with possibly more twisted field lines where the azimuthal component is larger relative to the radial component.

Physical Interpretation:

- The behavior of B_r and B_ϕ suggests that the dynamo is operating in a regime where both diffusion and possibly other dynamo processes like advection or wave propagation are important.
- The changes in the magnetic field components indicate complex interactions within the dynamo that could lead to the generation of magnetic fields through processes like the alpha effect or the omega effect in dynamo theory.

Exploring how different seed fields affect the results





Initial Seed Fields:

- The simulation starts with random initial conditions for both B_r and B_ϕ.
- These initial seed fields are likely small perturbations meant to mimic natural fluctuations in a physical system.

Temporal Evolution:

- As the simulation progresses, these seed fields evolve according to the dynamo equations, which include the effects of fluid motion, magnetic induction, and diffusion.

Growth and Saturation:

- Depending on the dynamo mechanism at play (e.g., the alpha effect, differential rotation, etc.), the magnetic field components can either grow (if the conditions are right for a dynamo action) or decay (if the dynamo is not sustained).
- Eventually, the fields may reach a saturated state where the growth is balanced by dissipative processes.

Spatial Distribution:

- The animation would show how B_r and B_ϕ vary across the radial distance from the center of the dynamo.
- This distribution is crucial for understanding the structure of the magnetic field and how it might influence the dynamo's operation.

Interaction Between Components:

- The interaction between B_r and B_ϕ is also of interest.
- In some dynamo models, the azimuthal field B_ϕ is generated from the radial field B_r through stretching by differential rotation, while B_r can be regenerated from B_ϕ through processes like the alpha effect.

Dependence on Seed Fields:

- The final magnetic field configuration can be highly dependent on the initial conditions.
- The simulation might show how different initial seed fields lead to different outcomes, illustrating the sensitivity of the dynamo process to initial conditions.

These results contatining the animated plots are in my personal webpage.Here is the link of my personal webpage.Ritabik Banerjee.

Navigate to Projects——>click on the All projects Option——>In term papers click on the plasma physics section to more details.

IV. CONCLUSION

The investigation into galactic dynamo processes through numerical simulations and differential equations has provided profound insights into the behavior of magnetic fields within galactic environments. The examination of the galactic magnetic field, influenced by the diffusion equation under varying turbulent magnetic diffusivity (η_t), has unveiled a nuanced understanding of its intricate evolution. The dynamo simulation, utilizing the Runge-Kutta method, showcased the temporal evolution of the radial magnetic field component (B_r), revealing non-monotonic patterns with intricate spatial variations. Exploring diverse scenarios involving varying initial conditions, boundary conditions, and seed fields,

the simulations emphasized the sensitivity of the dynamo process to these parameters. Magnetic field components (B_r and B_ϕ) exhibited intricate behaviors influenced by boundary conditions and initial seed fields, contributing to a deeper comprehension of the dynamo mechanism's sensitivity.
The analysis of pitch angle variation, temporal evolution, and radial dependence further elucidated the dynamics of field lines, emphasizing the importance of understanding the interplay between radial and azimuthal components. The inclusion of an exponential decay fit provided a quantitative measure of the decay rate, revealing the complex interplay between dynamo growth and dissipative processes.

Moreover, the investigation into the nonlinear dynamics of magnetic field components unveiled oscillatory patterns, reversals, and complex behaviors, underscoring the intricate nature of galactic dynamo systems. The spatial distribution of B_r and B_ϕ across radial distances provided crucial insights into the structural aspects of the magnetic field, influencing the dynamo's overall operation.
In conclusion, this comprehensive study, utilizing differential equations, numerical methods, and simulations, has significantly advanced our understanding of galactic dynamo processes, offering a valuable foundation for further research in astrophysics. The presented findings contribute to the broader knowledge of magnetic field evolution in galaxies and provide a basis for future investigations into the underlying physical mechanisms governing these intricate systems.

V. REFERENCES

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